JOURNAL OF ECONOMIC DEVELOPMENT 103
Volume 28, Number 2, December 2003

THE SOLOW'S MODEL WITH ENDOGENOUS POPULATION: A NEOCLASSICAL GROWTH CYCLE MODEL

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It is shown here that the Solow (1956) neo-classical growth paradigm not only explains the "first" stylised fact of economic growth, namely the existence of a globally stable state of balanced growth, but, once endowed with a demographically founded formulation of the labour supply, is also capable to endogenously explain a second main stylised fact of growth, i.e., the generation of globally stable oscillations around the path of balanced growth.

Keywords: Solow's Balanced Growth Model, Endogenous Population, Neoclassical Growth-Cycle Model
JEL classification: E3, J0

1. INTRODUCTION

The centrality of the neo-classical growth model of Solow (1956) for economic theory is witnessed by the current persistency of new contributions stimulated by his work (for instance Bajo-Rubio (2000)). The aim of the present paper is to investigate the effects of endogenous labour supply within Solow's model.

This endogenisation is based on a coupling of an age structure argument (absent in the original Solow's work) plus a Malthusian relation between fertility and income, which was well recognised by Solow himself. A realistic formulation of labour supply must take into account past demographic behaviours in that the new entries into the labour force at time t are the outcome of fertility behaviours of past generations. This requires the introduction of the age structure prevailing in the population. This fact was well known to the "classical" economists: "the supply of labourers in the market can neither be speedily increased when wage rise, nor speedily diminished when they fall.

* The authors thank an anonymous referee of this journal for several useful comments that greatly improved the quality of the paper. Usual disclaimers apply.

When wages rise a period of eighteen or twenty years must elapse before the stimulus, given the principle of population, can be felt in the market" (McCulloch (1854, p. 34)). The previous McCulloch's sentence contains, according to the current literature in population economics, a terse definition of Malthusian cycles. Malthusian cycles are a firm theoretical point in population economics. They are the consequence : "...of the lags between the response of fertility to current labour market conditions and the time when the resulting births actually enter the labour force" (Lee (1997, p. 1097)). The existence of such cycles was lucidly predicted, at least theoretically, by Malthus in several loci in his famous Essay (Malthus (1992)).

Long run oscillations of population and of the economy are an important, though sometimes neglected (for instance Barro and Sala i Martin (1995), disregard it), stylised fact of economics. They are well documented in the recent empirical study by Lee and Anderson (2002) (see in particular their figs. 1, 2, 3 reporting UK historical time series of the death and birth rates, and of the wage rate). They state: "Both series, especially the birth rate, show broad swings over hundreds of years, and short term fluctuations are substantial, especially prior to 1750" (ibidem, p. 201). Another fundamental example is represented by the so called post-transitional fertility fluctuations, such as the baby-booms observed in most of the Western World, sometimes explained by the Easterlin's effect (Easterlin (1961)). We finally feel that if Lowest Low Fertility, as recently observed in the Western World (Kohler *et al.* (2002)), should finally come to an end - and this is considered more or less a necessary event by many scholars (Bongaarts (1999)) - this would almost necessarily give rise to a new fertility wave.

In this paper the age structure of the population, which governs Malthusian cycles, is embedded in our Solow-type model by resorting, in a manner very close to the classical economists thought, to time-lags.[1] We show that persistent oscillations may occur in the Solow's model when the rate of change of the labour supply is correctly assumed to depend (even in the simplest manner) on past demographic behaviours. Thus we can argue that the neoclassical growth paradigm, e.g., the Solow's model, is capable not only to explain the stylised fact of balanced growth, but, once endowed with a correctly demographically founded formulation of the labour supply, also to endogenously explain a second main stylised fact of economic growth, namely the appearance of steady oscillations around the "average" path of balanced growth.

In order to position the present paper in the current literature on economic growth, we note that our model is a Solovian model of balanced growth with endogenous population. In this sense it may be considered an "endogenous" growth model, in that it

[1] The idea that time-delays represent a simple and clever strategy to embed age structure within complex models, dates back to Vito Volterra famous 1926 paper on interactions between biological species. For a modern presentation see McDonald (1978). This strategy is of course an approximation of the true underlying age structure model, based on the assumption of "stability" of the age composition, i.e., the constancy of the ratios between different cohorts of the underlying population.

endogenously explains the rate of total income growth.[2] Additionally, it obviously is a "descriptive" model, rather than a "fully optimized" one as those surveyed in Tamura (2000). Further, our model has the following features: 1) it is a growth cycle model; 2) it endogeneises the age structure of the population in a macroeconomic setting. The two latter features are not shared by most available growth models, whether they belong to the Solovian tradition or to the "new growth theory".

The present paper is organised as follows. In section two a basic Solow-type model with endogenous population depending on current fertility is introduced, and its properties are studied. In section three we introduce a more general model embedding a time-lag in the reaction of the rate of change of the labour supply to past income, and its properties are investigated by means of local stability analysis plus Hopf bifurcation. Section four completes the analysis of the model by investigating its global properties via numerical simulations and discussing the economic meaning of its main results. The "core" results are summarised in the conclusions.

2. A BASIC SOLOW'S MODEL WITH ENDOGENOUS POPULATION

The original Solow growth model (Solow (1956)) is defined by the ordinary differential equation (ODE):

$$\dot{k} = sf(k) - n_s k \, , \tag{1}$$

where $k = K/L$ denotes the capital-labour ratio, $y = f(k)$ the production per unit of labour, s the saving rate $(0 < s < 1)$, $n_s > 0$ the rate of change of the supply of labour $\dot{L}/L$, which was initially assumed exogenous by Solow. As in the Solow 1956 paper, we have assumed $\delta = 0$ where δ is the rate of capital depreciation. When a Cobb-Douglas production function is chosen (ibidem, example 2, p. 76), the model collapses into the ODE:

$$\dot{k} = sk^{\alpha} - n_s k \, , \tag{2}$$

where $0 < \alpha < 1$. Contrary to most subsequent developments, where the supply of labour was treated as exogenously determined, Solow also tried, still in his 1956 paper, to endogenise it. He wrote (*ibidem*, p. 90) the rate of change of the supply of labour as a function of the current level of per-capita income: $n_s = n_s(f(k))$, which becomes in the Cobb-Douglas case

[2] We note that conversely our model does not allow a positive rate of growth of the per-capita income growth, which is a major task of the "new growth theory".

$$n_s = n_s(k^\alpha)\,. \tag{3}$$

He also gave some qualitative analysis of the effects of a general non monotonic n_s function.[3]

As in this paper we are essentially interested in the effects of forces of "fundamental" nature, in what follows we assume, for simplicity, that the function n_s be linear and increasing, i.e., we consider $n_s = nk^\alpha$ where $n > 0$ is a constant parameter,[4] tuning the reaction of the rate of change of the labour supply to changes in per-capita income. We therefore have the model:

$$\dot{k} = sk^\alpha - nk^{1+\alpha}\,. \tag{4}$$

It is easy to see that Equation (4) always admits, as in the basic Solow's model, the zero equilibrium (E_0) and a unique positive equilibrium (E_1) which is globally asymptotically stable (the previous conclusions are preserved if we assume $n(\cdot)$ to be a general increasing function, saturating or not). In particular the capital-labour ratio at E_1 is $k = k_1 = s/n$.

It is worth noting that the long term growth rate of total income (Y), in model (4) is:

$$\frac{\dot{Y}}{Y} = \frac{\dot{y}}{y} + n_s = nk_1^\alpha = n^{1-\alpha}s^\alpha\,. \tag{5}$$

[3] For sake of precision, we notice that Solow considered a qualitatively complicated form of the fertility function, generating three equilibria, and also predicted, in a pioneering way, the possibility of endogenous self-sustained growth allowed by the endogeneity of labour supply. This idea will be better formalised many years later, for instance by Becker, Murphy and Tamura (1990).

[4] According to a purely demographic interpretation of labour supply, the positive relationship between labour supply and wage adopted here may be interpreted as a Malthusian relation between fertility and the wage, rather than a "well behaved" labour supply function. We did not explicitly considered here income-related mortality but the extension is straightforward.

Moreover, referring to Malthus' original ideas, the rate of change of labor supply could also be specified as $\dot{L}/L = n(f(k) - y°)$, where $y°$ is subsistence income and $n > 0$. In this specification, the growth of the labor supply would be negative when per capita income is below the subsistence level, which is a realistic fact. But also this specification (as well as any non linear saturating specifications) leads to the same results presented here under the simpler specification adopted above.

Thus the long term growth rate of total output depends on the joint action of population (as summarised through the Malthusian parameter n) and saving. The interest for this result lies in the relevance of its message, and in the simplicity with which it is obtained: contrary to Solow's original result, where, paradoxically, saving determines the accumulation but does not play any role for long term growth of total income, saving matters for growth. This is widely reasonable, as it is only through saving, e.g., new investments, that population change may promote economic growth.

Clearly, the assumption that the rate of change of the supply of labour is a function of current income, as in (4), might perhaps reflect an underlying participation effect, but is hardly defendable as determined by true population change. We argue, however, that the simple model (4) is useful to cast the investigation of the consequences of fully endogenous population dynamics, as will be done in the next section by resorting to time-lags.

3. AGE STRUCTURE AND LABOUR SUPPLY: THE DEMOGRAPHIC DELAY

3.1. Motivations

To take into account of the overall demographic mechanism of age structure in economic models would lead to a larger dimension and thus more analytical complexity. Since the mathematical literature (Mac Donald (1978)) has shown that time-delays often represent simple approximations of age structure mechanisms, in this paper we follow this route. The intuitive idea here is that the current rate of change of the supply of labour is related to past fertility, and thus to past levels of the wage, following a prescribed pattern of delay. There are two main alternatives: fixed delays and distributed delays. The former is better suited when there is no variability in the process of transmission of the past into the future: for instance when we assume that all individuals are recruited in the labour force at the same fixed age. Conversely when recruitment may occur at different ages, i.e., with different delays (for instance because the time needed to complete formal education is heterogeneous within the population), distributed delays appear more suitable.[5] The introduction of a distributed delay in the population term in (4) leads to the following integro-differential (IDE) equation:

$$\dot{k} = sk^{\alpha} - \left(\int_{-\infty}^{t} n\big(k^{\alpha}(\tau)\big) G(t-\tau) d\tau \right) k \,, \tag{6}$$

[5] See Invernizzi and Medio (1991) for an economically oriented discussion of distributed delays.

where the term $n(k^{\alpha}(\tau)),\ \tau < t$ captures past (rather than current), income-related fertility, and $G(t-\tau)$ is the corresponding delaying kernel. As well known (MacDonald (1978)), by assuming that the G kernel belongs to the so-called Erlangian family,[6] IDEs as (6) may be reduced to higher order systems of ordinary differential equations.

3.2. The Model: Equilibrium and Dynamical Analysis

In economics the most frequent type of delaying kernel is based on the notion of exponentially fading memory (obtained for $r=1$ in the Erlangian family). However an exponentially fading memory, which implies to give maximal weight to the "more recent" past, cannot be considered a reasonable representation of the demographic mechanism. Rather the new entries in the labour supply at each moment t of time are due to individuals born in a rather narrow interval of time in the past. We argue that "humped" memories, rather than exponential ones, give a satisfactory representation of the demographic process (i.e., past fertility plus delayed transition into the labour force). We therefore choose as the delaying kernel the second member of the Erlangian family $(r=2)$:

$$G(x) = Erlang_{2,\beta}(x) = \beta^2 x e^{-\beta x}, \qquad x>0\ ;\ \beta>0, \tag{7}$$

because it is the simplest type of "humped" Erlangian density (having a mean delay $T = 2/\beta$). This implies that the maximal weight is given to some appropriate time interval in the past, which heuristically well corresponds to the idea of demographic delay.[7] Under (7) the IDE (6) may be reduced to an ODE system by the linear chain trick. This is obtained, in our case, by introducing $r=2$ auxiliary variables, the first one of which is the integral term in (6) where $G = Erlang_{2,\beta}(x)$, and the second one is obtained by replacing, in the integral term, $Erlang_{2,\beta}(x)$ with $Erlang_{1,\beta}(x)$. We obtain, after the further simplifying change of variable $k^{\alpha} = Z$, the 3-dimensional ODE system:

[6] A density function $f(x)$ is erlangian with parameters (r,β) when it has the form:

$$f(x;r,\beta) = \frac{\beta^r}{(r-1)!} x^{r-1} e^{-\beta x}, \quad x>0\ ;\ r=1,2,...\ ;\ \beta>0.$$

[7] The use of a "more humped" distribution in the Erlangian family (as erlangian densities of higher order, i.e., $r = 3,4,5...$), which could perhaps more realistically capture the phenomenon of the delayed entry into the labour force, simply confirms the more important results of this paper (e.g., the onset of persistent oscillations as shown in next section). We did not investigate analytically the properties of higher order systems with higher order kernels because they become more and more analytically cumbersome without offering any extra "qualitative" insight from the economic point of view.

$$\dot{Z} = \alpha Z\left(sZ^{\frac{\alpha-1}{\alpha}} - nX\right),$$
$$\dot{X} = \beta(R - X), \tag{8}$$
$$\dot{R} = \beta(Z - R).$$

In a nutshell the linear trick procedure implies r additional linear differential equations. In particular every additional equation represents a continuous time adaptive mechanism, as evident from (8). For technical details on the linear chain trick see Mac Donald (1978), Fanti and Manfredi (1998).

Model (8) has the zero equilibrium E_0 and the positive equilibrium $E_1 = (Z_1, X_1, R_1)$ where:

$$Z_1 = X_1 = R_1 = \left(\frac{s}{n}\right)^{\alpha}. \tag{9}$$

Thus, besides the change of variable $k^{\alpha} = Z$, model (8) preserves the equilibria of its unlagged counterpart (4), so that Equation (5) holds with its interesting economic interpretation.

The local stability analysis of the positive equilibrium E_1 gives the third order characteristic polynomial: $P(X) = X^3 + a_1 X^2 + a_2 X + a_3$ with coefficients:

$$a_1 = 2\beta + (1-\alpha)nZ_1, \quad a_2 = 2\beta(1-\alpha)nZ_1 + \beta^2, \quad a_3 = \beta^2 nZ_1,$$

which are all strictly positive. Therefore, by the Routh-Hurwitz stability test, E_1 will be locally stable provided the condition $\Delta_2 = a_1 a_2 - a_3 > 0$ holds. This leads to the stability condition:

$$f(\beta) = 2\beta^2 + nZ_1(4-5\alpha)\beta + 2((1-\alpha)nZ_1)^2 > 0. \tag{10}$$

The discussion of (10) shows that: i) for $\alpha < \alpha_2 = 0.88$ no loss of stability is possible; ii) for $\alpha > \alpha_2$, $f(\beta)$ always has two strictly positive real roots, β_1, β_2 (with

$\beta_1 < \beta_2$), implying that losses of stability may occur. In particular both β_1, β_2 values represent Hopf bifurcation values of the β parameter.[8] They are given by:

$$\beta_{1,2} = \frac{n^{1-\alpha}}{4} s^{\alpha} \left((5\alpha - 4) \pm \sqrt{(9\alpha - 8)\alpha} \right) \quad . \tag{11}$$

The latter formula provides a nice relation between the average demographic delay (we recall that the mean delay is $T = 2/\beta$ and the economic parameters.

We may summarise our main findings by the following:

Proposition 1: When the profit share α is below a prescribed threshold value ($\alpha < \alpha_2$), system (8) replies the behaviour of the original Solow's model, with convergence to the unique and globally stable equilibrium E_1. When $\alpha > \alpha_2$, system (8) continues to converge to the globally stable equilibrium E_1 only when β is sufficiently large or sufficiently small, i.e., for $\beta > \beta_2$, or $\beta < \beta_1$. In the whole window $\beta_1 < \beta < \beta_2$ the E_1 equilibrium is locally unstable. At the points $\beta = \beta_1$, $\beta = \beta_2$, Hopf bifurcations occur.

In words: an endogenous labour supply according to the demographic mechanism adopted here is able to destabilise the equilibrium of the Solow's model and to generate persistent oscillations[9] via Hopf bifurcations.

4. SIMULATIVE EVIDENCE AND WORKING OF THE SYSTEM

The fact to know that a Hopf bifurcation exists nothing says about the stability properties of the involved periodic orbits, e.g., it does not say whether the bifurcation is supercritical or subcritical (that is, whether the periodic orbit is locally stable or unstable). Unfortunately the investigation of stability of periodic orbits emerged via

[8] The formal proof requires to show that (Marsden and McCracken, (1976)) : i) purely imaginary eigenvalues exist for the linearised system at $\beta = \beta_1$ or $\beta = \beta_2$ due to a "continuous" movement of a pair of complex eigenvalues; ii) the crossing of the imaginary axis by the involved complex pair occurs with nonzero speed. The proof, omitted here for brevity, is available on request.

[9] Of course this is not the only avenue through which to obtain population induced persistent oscillations in a neoclassical growth model: for instance, as suggested by a referee of this Journal, this could be obtained by a two-period overlapping generation model - rather than the Solovian one developed in our paper - including a model of adaptive expectation on fertility determination.

Hopf bifurcation at dimensions greater than the second is a hard task (Marsden and McCracken (1976)). Moreover the predictions of the Hopf theorem are "local": they nothing say about global behaviours.

We therefore resorted to numerical simulation to clarify the stability nature of the Hopf bifurcations occurred at the points $\beta = \beta_1$, or $\beta = \beta_2$ and more generally to investigate the global properties of our model.

Let us now summarise our main dynamical findings and illustrate the working of the model. There is a region, defined by $\alpha < \alpha_2 = 0.88$, in which the traditional behaviour of Solow's model is confirmed: the economy converges to a long term globally stable steady state. Conversely, for very large $\alpha (\alpha > \alpha_2)$ the economy may be destabilised by the action of the demographic delay. More in detail, as long as β is very large (in relative terms), i.e., for $\beta > \beta_2$ (we recall that this means "very" small mean demographic delays), the E_1 equilibrium preserves its stability. But as β is decreased (this happens for increasing mean delays) stability may be lost. This happens for $\beta = \beta_2$ where a first Hopf bifurcation occurs and E_1 exchanges its stability with a stable limit cycle. Finally, by further decreasing β, a further bifurcation occurs at $\beta = \beta_1$ where the local stability of the E_1 equilibrium is restored. Therefore for very large mean delays the Solow's behaviour is restored once again. Table 1 reports a synoptic view of the process of phase transition of the positive equilibrium when $\alpha > \alpha_2$.[10]

Table 1. Windows of the delay parameter β and the corresponding behaviour of the model as summarised by the nature of its unique positive equilibrium E_1.

□ β	$(0, \beta_1)$	□ $[\beta_1, \beta_2)$	$[\beta_2, +\infty)$
E_1 is :	stable (node or focus)	unstable (surrounded by a stable limit cycle)	stable (node or focus)

It is worth to remark the following further facts suggested by simulation: i) both the points $\beta = \beta_1$, $\beta = \beta_2$, generate supercritical bifurcations (i.e., locally stable oscillations); ii) the whole window $\beta_1 < \beta < \beta_2$ is a region of stable oscillations, iii) all the

[10] We notice that, although persistent oscillations seem to require a rather large value of the elasticity of capital α (even higher than those estimated by using a broad definition of capital stock including the human capital), this is just a feature of the low-order kernel ($r = 2$) considered here for purposes of analytical simplicity. Indeed by resorting to higher order distributions of the delaying kernel (e.g., erlangian densities of order $r = 3,4,5...$), which are more realistic, we obtained, via simulation, more realistic "critical" values of the elasticity of capital.

properties of the model seem to hold globally: when the E_1 equilibrium is either locally stable, or surrounded by locally stable limit cycles, the system always seems globally stable.

Thus our results are consistent with two important stylized facts: 1) the existence of global stability of the economy, thus confirming Solow's result; 2) the presence of (stably) fluctuating behaviours around the state of balanced growth of the economy. The latter fact greatly enriches Solow's result.

Figures 1a and 1b report a two-dimensional view of the emerging cycles respectively at the largest bifurcation point $\beta_2 = 0.064$, and at the smaller one $\beta_1 = 0.0056$ (other parameter values: $\alpha = 0.92$, $s = 0.3$, $n = 0.01$).

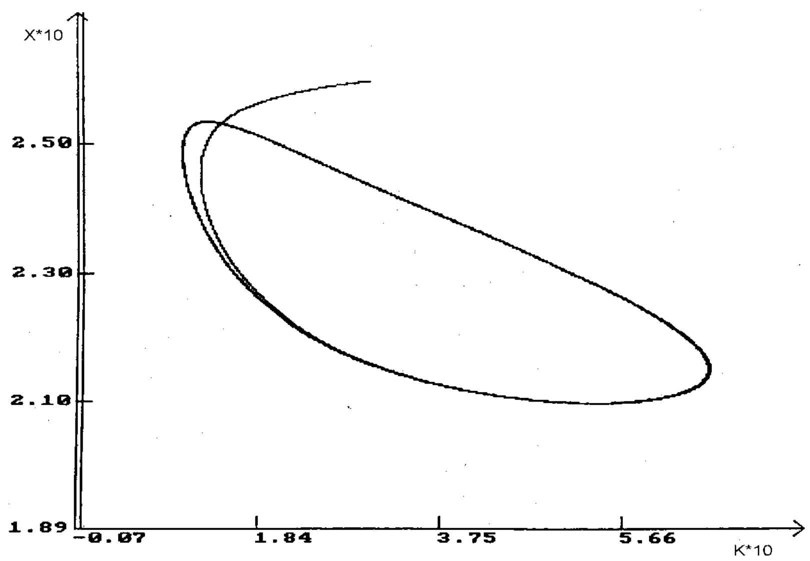

Figure 1a. A Stable Limit Cycle appeared at $\beta = \beta_2 = 0.064$

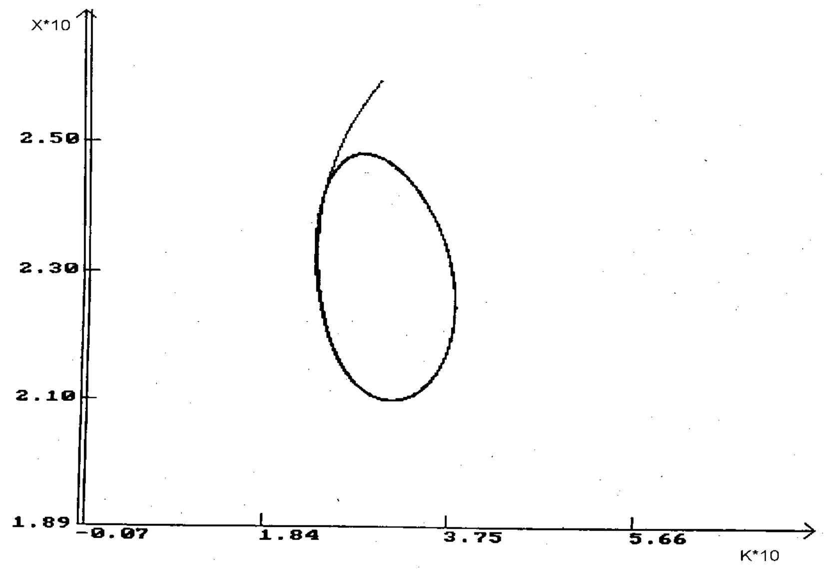

Figure 1b. A Stable Limit Cycle appeared at $\beta = \beta_1 = 0.0056$

The economic interpretation of our findings is straightforward: when returns are "not too decreasing" (i.e., $\alpha > \alpha_2$), an increase in the age of entrance in the labour force may destabilise the equilibrium of the Solow's model, and trigger persistent oscillations. These latter appear to be bounded, so that the global stability of the economy is preserved. These fluctuations appear to be the outcome of the demographic memory operating through the age structure delay. The fact that periodic behaviours may persist also for very long mean delays seems to suggest the existence of possible "supergenerational" echoes, not necessarily related only to past fertility but to other long lasting delayed effects.

5. CONCLUSIONS

In this paper we have shown that the Solow's model, once extended to take account for: i) the existence of a delay in the process of recruitment in the labour force, due to the age structure of the population, ii) the existence of a Malthusian relation between fertility and wage, is capable of generating stably fluctuating growth paths. This fact shows that the neoclassical growth paradigm not only explains the stylised fact of balanced growth, but, once endowed with a correctly demographically founded formulation of the labour supply, becomes capable to endogenously explain the other main stylised fact of economic growth, namely the generation of globally stable oscillations around a path of balanced growth. An interesting consequence of the presence of endogenous population is that, contrary to Solow's original result, where, as well known, saving determines the accumulation but does not play any role for long term growth, saving matters for (long term) growth. The main message is thus that it is only through savings, (and therefore investments), that population growth may eventually promote economic growth.

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Manuscript received December, 2002; final revision received July, 2003.