

# A Modified Version of Breeden's Capital Asset Pricing Model

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The market-based model, developed by Sharpe and Lintner, asserts that the risk premium is proportional to the covariance of the rate of return with the rate of return on the market portfolio. Recently Breeden has put forward a consumption-based model as an alternative: he replaces the rate of return on the market portfolio with aggregate consumption.

Using the permanent-income theory of consumption demand, we modify the Breeden's model as a permanent-consumption based CAPM. We show that the rate of return on the market portfolio must be replaced by the permanent consumption, not by the actual consumption.

## I. Introduction

The market-based model, developed by Sharpe and Lintner, asserts that the risk premium is proportional to the covariance of the rate of return with the rate of return on the market portfolio. Recently Breeden has put forward a consumption-based model as an alternative: he replaces the rate of return on the market portfolio with aggregate consumption.

Many investigators have tested the standard CAPM, but they have furnished little econometric evidence that the model is actually valid. Roll argues that the econometric tests of the traditional model prove little because one cannot accurately measure the rate of return on the market portfolio. Breeden argues that

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consumption-based model provides a testable alternative to the traditional model, because there exists extensive data on consumption.

But in his derivation, Breeden does not distinguish between actual consumption and permanent consumption. The purpose of this paper is to show, using the permanent-income theory of consumption demand, why the change of wealth must be replaced by the change of permanent consumption, not by the change of actual consumption.

This paper is organized as follows. In section II, we discuss the theoretical analysis of securities market equilibrium in a continuous-time framework. We first derive the market-based CAPM. Next we derive Breeden's consumption-based capital-asset pricing model. In section III, working with the permanent-income theory of consumption demand, we modify the Breeden's model as a permanent consumption based capital-asset pricing model. Finally, we summarize the findings and describe further directions of research.

## II. Portfolio Choice in Continuous-Time

Let us first discuss the analytical assumptions. We assume that there are no transactions costs or taxes in the economy. All assets are infinitely divisible. At every instant there is a price for each asset at which all individuals and firms may buy and sell the asset. There are no institutional restrictions; that is, a short sale of any asset is allowed, with full use of the proceeds. The log of the price of each security in the market follows a stationary Brownian motion and trading in assets takes place continuously in time. Individuals and the managers of the firms act as if their actions do not affect prices. We assume that a competitive equilibrium exists and we do not consider whether the equilibrium is unique or stable. We are interested in the characteristics of equilibrium.

It is assumed that there are a riskless asset and  $n$  risky assets. The rate of return on the riskless asset is  $r$ . The riskless asset is instantaneously riskless; at each instant of time, each investor knows with certainty that he can earn rate of return  $r$  over the next instant by holding the asset. The vector of the rates of return on the risky assets is  $d_a$ , which includes dividends as well as capital gains. We

assume that the instantaneous rate of return  $da$  is generated by an Itô process;

$$(1) da = \mu dt + dz.$$

Here  $\mu$  is an  $n$ -vector;  $dz$  is an  $n$ -vector of stationary Wiener processes such that with  $E_t(dz) = 0$  and  $\text{Var}_t(dz) = S$ . The instantaneous rates of return vector thus has mean  $\mu$  and variance  $S$ .

Given his wealth, at each instant each individual chooses an optimal rate of consumption and optimal portfolio. Wealth is  $w$  and consumption is  $c$ . The  $n$ -vector  $f$  describes the portfolio. The component  $f_i$  denotes the fraction of the total wealth invested in the risky asset  $i$ . The remainder of the wealth is invested in the riskless asset. Given this framework, the investor's wealth change can be given by the stochastic differential equation;

$$(2) dw = (1 - f'l_n) wr dt + f'w da - c dt.$$

Here  $(1 - f'l_n)$  is the fraction of the total wealth invested in riskless asset;  $l_n$  is an  $n$ -vector of ones. Substituting (1) for  $da$ , we can rewrite (2) as

$$(3) dw = (f'(\mu - rl_n) + r)w dt + wf' dz - c dt.$$

The choice of  $f = (f_1, f_2, f_3, \dots, f_n)'$  is unconstrained, because the fraction  $(1 - f'l_n)$  of the total wealth is invested in riskless asset.

We assume that each individual  $k$  acts to maximize the expected value of his time-additive, state-independent, strictly concave *Von Neumann-Morgenstern* utility function;

$$(4) \max_{c, f} E_t \left\{ \int_t^{\infty} U[c(\tau), \tau] d\tau \right\},$$

subject to the budget constraint (3). Here  $E_t$  is the conditional expectation operator, conditional on the information at time  $t$ .

We can derive the necessary optimality conditions for an investor facing the problem (4). First let us define the valuation function  $V[w(t), t]$  as follows;

$$(5) V[w(t), t] = \max_{c, f} E_t \left\{ \int_t^{\infty} U[c(\tau), \tau] d\tau \right\}.$$

By Bellman's principal of optimality (see Chow), maximizing utility (4) is equivalent to solving the following functional equation;

$$(6) 0 = \max_{c, f} U(c, t) + V_t + V_w \{ [f'(\mu - r1_n) + r]w - c \} +$$

$$(1/2)V_{ww} w^2 f' S f.$$

where all subscripts of  $V$  denote the partial derivatives, and  $S$  is the  $n \times n$  instantaneous variance-covariance matrix between the returns.

The  $n + 1$  first order necessary conditions for an interior maximization in (6) are obtained by differentiating with respect to the  $c$ ,

$$(7) 0 = U_c - V_w$$

and differentiating with respect to the  $f$ ,

$$(8) 0 = V_w (\mu - r1_n) + V_{ww} w S f.$$

Since  $U[c(t), t]$  is concave, the second-order conditions will be satisfied.

Equation (7) is the usual envelope condition to equate the marginal utility of current consumption to the marginal utility of wealth (future consumption). The manifest characteristic of (8) is its linearity in the portfolio demands; hence we can solve explicitly for these functions;

$$(9) f = (-V_w / w V_{ww}) S^{-1} (\mu - r1_n).$$

Define  $S^{-1} (\mu - r1_n) / 1_n' S^{-1} (\mu - r1_n)$  as the "efficient portfolio of risky assets." Define  $R_R = -(w V_{ww} / V_w)$ , the relative risk aversion. Equation (9) shows that the individual invests partly in the efficient portfolio of risky assets and partly in the riskless asset.

The proportion invested in each is determined by the relative risk aversion. The investment in the efficient portfolio of risky assets decreases as the risk aversion increases.

### A. Market-Based CAPM

Now we consider market equilibrium. Let the superscript  $k$  denote the value for individual  $k$ . The total demand for risky assets is  $\sum_k f^k w^k$ . We assume homogeneous expectations; all individuals have the same expectations.

The market portfolio refers to the portfolio of all outstanding assets. The  $n$ -vector  $f$  describes the market portfolio;  $f_i$  is the fraction of the total wealth in asset  $i$ . Consequently,  $1 - f'1_n$  is the fraction of total wealth in the riskless asset. The rate of return on the market portfolio is

$$\begin{aligned} (10) \quad da_m &= f' da + (1 - f'1_n)r dt \\ &= (f'(\mu - r1_n) + r)dt + f' dz. \end{aligned}$$

The market rate of return has mean  $f'(\mu - r1_n) + r$  and variance  $f'Sf$ .

Equilibrium requires that supply equals demand:

$$(11) \quad fw = \sum_k f^k w^k.$$

Hence substituting  $f^k$  from (9) gives the market portfolio:

$$(12) \quad f = \left[ \sum_k (1/R_R^k) w^k / w \right] S^{-1} (\mu - r1_n) = (1/R_R) S^{-1} (\mu - r1_n),$$

in which we define  $1/R_R = \sum_k (1/R_R^k) w^k / w$ . Thus  $R_R$  is the harmonic mean of the relative risk aversion across individuals in the sense that  $R_R$  is the weighted average of individual relative risk aversion with weights  $w^k$ . Equivalently,

$$(13) \quad \mu = r1_n + R_R Sf = r1_n + R_R \text{Cov}_t(da, da_m).$$

where  $\text{Cov}_t(da, da_m)$  is the  $n$ -vector of the covariance of the asset returns with the market rate of return. The expected value and the covariance are conditional on all information available at the time  $t$ .

The standard capital asset pricing model (13) derived in the continuous-time framework (Merton) is same as the model developed by Sharpe and Lintner. The model states that the risk premium is proportional to the covariance of the rate of return with the rate of return on the market portfolio.

### B. Consumption-Based CAPM

Recent advances in theoretical finance have suggested a fresh way of looking at our limited data. One of the advances is the consumption-based capital-asset pricing model of Breeden. He restates the standard CAPM as a consumption-based model. Using the model presented above, we derive the consumption-based CAPM (Breeden's derivation is more general, in that he allows  $\mu$  and  $S$  to be stochastic).

The key property used in the derivation is the first-order necessary condition (7). Optimum consumption is a function of wealth and time,  $c(w, t)$ . Differentiating (7) with respect to the  $w$  gives  $U_{cc}c_w = V_{ww}$ .

Therefore

$$(14) \quad -V_w/V_{ww} = -U_c/(U_{cc}c_w).$$

Substituting (14) into (12) yields

$$(15) \quad fw = -(U_c/U_{cc}c_w) S^{-1} (\mu - r1_n).$$

<sup>1</sup> Premultiply (13) by  $f'$ , we have

$$(13)' f'\mu = f'rl_n + R_R f'Sf.$$

Since the market rate of return  $\mu_m$  has mean  $f'(\mu - r1_n) + r$  and variance  $\text{Var}_t(da_m)$  is  $f'Sf$  by (10), we can rewrite (13)' as following:

$$\mu_m = r + R_R \text{Var}_t(da_m). \text{ So we have } R_R = \frac{\mu_m - r}{\text{Var}_t(da_m)}$$

Define  $\beta = \frac{\text{Cov}_t(da, da_m)}{\text{Var}_t(da_m)}$ , then we can rewrite (13) as  $\mu - r1_n = \beta(\mu_m - r)$ .

Premultiplying (15) by  $S$  and using the  $\text{Cov}_t(da, dw) = wf'S$  by (1) and (3), we can rewrite (15) as

$$(16) \mu - r1_n = -(U_{cc}c_w/U_c)\text{Cov}_t(da, dw),$$

where  $\text{Cov}_t(da, dw)$  is the  $n$ -vector of covariances of asset returns with the wealth change. Since the optimal consumption is a function of wealth and time  $c(w, t)$ , Itô's lemma implies  $dc = c_w dw + c_t dt + (1/2)c_{ww} (dw)^2$ .

Therefore

$$(17) \text{Cov}_t(da, dc) = \text{Cov}_t(da, dw)c_w.$$

Intuitively (17) can also be seen by noting that the random change in consumption rate is linear in the random change in wealth, with the weight in the linear relation being the partial derivatives of the consumption with respect to the wealth. Thus the covariance of asset returns with the change in consumption can be expressed by (17). We rewrite (16) as

$$(18) \mu - r1_n = -(U_{cc}/U_c) \text{Cov}_t(da, dc).$$

We can readily aggregate this relationship. Let  $c = \sum_k c^k$  denote aggregate consumption. We have

$$(19) \text{Cov}_t(da, dc^k) = -(U_c/U_{cc}) (\mu - r1_n).$$

Summing across all individuals,

$$(20) \text{Cov}_t(da, dc) = -[\sum_k (U_c/U_{cc})^k] (\mu - r1_n).$$

So

$$(21) \begin{aligned} \mu - r1_n &= [-1/\sum_k (U_c/U_{cc})^k] \text{Cov}_t(dt, dc) \\ &= [-c/\sum_k c^k (U_c/cU_{cc})^k] \text{Cov}_t(da, dc/c). \end{aligned}$$

Thus  $-c/\sum_k c^k (U_c/cU_{cc})^k$  is harmonic mean of  $-c^k U_{cc}^k / U_c^k$ .

Therefore

$$(22) \mu - r1_n = -(cU_{cc}/U_c) \text{Cov}_t(da, dc/c),$$

or equivalently,

$$(23) \mu - r\mathbf{1}_n = -(cU_{cc}/U_c) \text{Cov}_t(da, \text{dln } c).$$

Here  $\text{Cov}_t(da, \text{dln } c)$  is  $n$ -vector of covariances of asset returns with the change in actual consumption and  $-cU_{cc}/U_c$  denotes the relative risk aversion. The model (22) and (23) say that risk premium depends on the covariance of the rate of return with the change in aggregate consumption.

Note that the derivation of this consumption-based CAPM is based on the first order condition  $U_c = V_w$ , which requires an additive utility function. Actually Grossman and Shiller derive the same result as Breeden. We have derived Breeden's result under the assumption that  $\mu$  and  $S$  are known, but Breeden shows that the same result holds if  $\mu$  and  $S$  are stochastic.

This consumption-based capital-asset pricing model (22) provides an empirically tractable framework for examining the interaction between asset returns and macroeconomy. Especially our ability to measure consumption can potentially circumvent the problem of explicitly identifying the market proxies, a problem discussed by Roll. Breeden suggests that

'The principal virtue of aggregate consumption measures, in comparison with the market proxies used, is that the consumption measures available cover of a greater fraction of the true consumption variable than the fraction that the market portfolio measures cover of the true market portfolio...'

In this part we have seen how Breeden's consumption-based capital-asset pricing model is derived from the standard CAPM in the continuous-time framework. Breeden's model is potentially empirically testable.

### III. The Permanent Consumption-Based CAPM

In this part we will present the permanent-consumption form as an alternative to the traditional CAPM. In deriving his result, Breeden does not distinguish between actual consumption and permanent consumption. The life-cycle, permanent-income hypothesis is widely accepted as the application of theory of the



consumer to the problem of dividing consumption between the present and the future. According to this hypothesis, consumers form estimates of their ability to consume in the long run and then set permanent consumption to their appropriate fraction of that estimate. The estimate may be stated in the form of wealth, following Modigliani, or permanent income, following Friedman. Therefore we have

$$(24) c_p \propto w,$$

where  $c_p$  denotes the permanent consumption and  $w$  is the wealth.<sup>2</sup>

Since stochastic parts of  $da_m$  and  $dw/w$  are the same, we have  $\text{Cov}_t(da, da_m) = \text{Cov}_t(da, dw/w)$ . Also, by (24), we have  $\text{Cov}_t(da, dw/w) = \text{Cov}_t(da, dc_p/c_p)$ . Using these, we can rewrite (13) as;

$$\begin{aligned} (25) \mu - r_n &= R_R \text{Cov}_t(da, dc_p/c_p) \\ &= R_R \text{Cov}_t(da, \text{dln } c_p), \end{aligned}$$

where  $\text{dln } c_p$  denotes the change in permanent consumption. In the permanent consumption-based model, the permanent consumption serves as a sufficient statistic for the underlying wealth. The risk premium depends on the covariance of the rate of return with permanent consumption.

#### IV. Summary and Further work

Recent advances in theoretical finance have suggested a fresh way of looking at the relationship between the expected return and risk. Especially Breeden restates the market-based model as a consumption-based model. We show how Breeden does develop his model in a continuous-time framework. Also we show that his

<sup>2</sup> For instance, it will be reasonable assumption that investors take time to adjust the change of wealth. Then the change of wealth can affect the future consumption as well as current consumption. In that case, we can make hypothesis that the change in permanent consumption may be expressed as a distributed-lead function of the current and future changes in actual consumption. Therefore the change of wealth can be replaced by the change of permanent consumption, as implied by (24).

derivation from the standard model is based on the fact that the change of wealth can be replaced by the change of aggregate consumption.

But in his derivation, Breeden does not distinguish between actual consumption and permanent consumption. In this paper, working with the permanent-income theory, we explain why the change of wealth will be highly correlated with the change of permanent consumption, not with the change of actual consumption.

Our permanent consumption-based model also enables the testing of the capital-asset pricing model, while avoiding Roll's criticism. But this test avoids the problem of measuring the rate of return on the market portfolio and replaces it with the problem of measuring the unexpected change in permanent consumption. The problem is subtle because one observes only the total consumption over a time interval, not the flow of consumption at each instant. If we can measure the change in permanent consumption, our modified model will provide an empirically tractable framework for examining the interaction between asset returns and macroeconomy.

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